

MAPPING CLASS GROUPS OF HEEGAARD SPLITTINGS OF SURFACE BUNDLES

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ABSTRACT. Every surface bundle with genus g fiber has a canonical Heegaard splitting of genus $2g+1$. We classify the mapping class groups of such Heegaard splittings in the case when the surface bundle has a sufficiently complicated monodromy map.

Let M be a closed surface bundle with bundle map $\pi : M \rightarrow S^1$. Equivalently, we can write $M = F' \times [0, 1]/\phi$ where F' is a closed surface of genus g and $\phi : F' \times \{0\} \rightarrow F' \times \{1\}$ is the *monodromy* of π . Assume $g \geq 2$. The complexity of the monodromy map can be measured by its displacement distance $d(\phi)$ in the curve complex for F' , whose definition we review below.

Every surface bundle M has a Heegaard surface Σ constructed as follows: Let $F = F_1 \cup F_2 = \pi^{-1}\{0, \frac{1}{2}\}$ be two fibers of the bundle. Let α_-, α_+ be vertical arcs with disjoint endpoints in F_1, F_2 , one in each component of $M \setminus F$. Construct Σ by attaching disjoint tubes to F along these arcs, as in Figure 1.

The *mapping class group* $Mod(M, \Sigma)$ is the group of automorphisms of M that take Σ onto itself, modulo isotopies of M that preserve Σ setwise. The *isotopy subgroup* of $Mod(M, \Sigma)$ is the set of elements that are isotopy trivial as automorphisms of M . In this short note, we define a subgroup $\mathcal{B} \subset Isot(M, \Sigma)$ for the standard Heegaard splitting of any surface bundle, then prove that for sufficiently high distance surface bundles, this subgroup is the entire isotopy subgroup of the mapping class group.

1. Theorem. *If the monodromy ϕ of π has displacement $d(\phi) > 28$ then $Isot(M, \Sigma) = \mathcal{B}$.*

The entire mapping class group $Mod(M, \Sigma)$ can often be deduced from $Isot(M, \Sigma)$ given further information about the automorphisms of M and its different genus $2g+1$ Heegaard splittings. However, that is beyond the scope of the present paper.

Note that the bound on $d(\phi)$ is independent of the genus of F . Bachman-Schleimer [2] showed that if g is the genus of F and $d(\phi) > 4g$ then Σ is a minimal genus Heegaard surface for M . However, for $g > 7$, it is conceivable

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that Σ is not the smallest Heegaard splitting for M (though we will see that it is unstabilized).

To define the displacement $d(\phi)$, recall that the *curve complex* $\mathcal{C}(F)$ for F is the simplicial complex whose vertices are isotopy classes of simple closed curves in F and whose simplices span sets of vertices representing pairwise disjoint curves. The map ϕ defines an isometry of $\mathcal{C}(F)$, which we will also denote by ϕ . The *displacement* of ϕ is $d(\phi) = \min\{d(v, \phi(v))\}$ where $d(\cdot, \cdot)$ is the edge-path distance between two vertices in the curve complex.

There is a pair of disjoint, weak reducing disks D^-, D^+ for Σ dual to the arcs α_-, α_+ , respectively, such that compressing along these disks recovers F . Note that any isotopy of α_{\pm} that keeps the arcs vertical with their endpoints disjoint and in F and returns to the original arcs will define an element of $Isot(M, \Sigma)$. We will call the subgroup of all such automorphisms the *arc subgroup* $\mathcal{A} \subset Isot(M, \Sigma)$.

The movement of the endpoints of α_1, α_2 in F_1 under such an isotopy defines a braid in F_1 , and thus determines a homomorphism $b : \mathcal{A} \rightarrow B_{2, F_1}$, where B_{2, F_1} is the pure braid group of two strands in F_1 . There is a second homomorphism from the kernel of b into B_{2, F_2} . In fact, the reader can check that this homomorphism is onto the subgroup of B_{2, F_2} in which the path of each strand is homotopy trivial. After conjugating by an isotopy that takes the path of one strand to the trivial path, we can identify this with the kernel of the inclusion map from the fundamental group of a once-punctured surface to a closed surface. We will show below that this homomorphism is injective, so \mathcal{A} is finitely generated and we can explicitly construct a generating set from the above description. Note that all the groups described above are finitely generated.

The bundle structure defines an infinite cyclic cover \hat{M} of M such that \hat{M} is homeomorphic to $F' \times \mathbf{R}$ and $\pi : M \rightarrow S^1$ lifts to the projection map of \hat{M} onto the \mathbf{R} factor of the product. We can assume the product structure is such that translating along the \mathbf{R} factor permutes the preimages in \hat{M} of points in M and thus defines an automorphism of M . All these automorphisms are isotopy trivial on M and translation by any half integer defines an automorphism $r_t : M \rightarrow M$ that takes F onto itself. (The automorphism will switch F_1 and F_2 whenever t is not a whole integer.) Because the arcs α_- and α_+ are contained in disjoint balls, we can choose the product structure on \hat{M} so that the automorphism induced by each half integer translation permutes α_- and α_+ and, moreover, takes Σ onto itself.

Note that this r_t is not unique – it is only defined up to composition with elements of \mathcal{A} . The different possible choices of r_t for a given t thus define a coset of \mathcal{A} . Define \mathcal{B} as the union of all such cosets for all half integers. By construction, this is an extension of \mathcal{A} by \mathbf{Z} . The subgroup \mathcal{A} is normal in \mathcal{B} , so we have a semi-direct product $\mathcal{B} = \mathcal{A} \rtimes \mathbf{Z}$, where \mathbf{Z} acts on \mathcal{A} by conjugating by a power of the monodromy.

The subgroup \mathcal{A} consists entirely of reducible automorphisms of Σ . However, it does not fall into the classification of reducible automorphisms of strongly irreducible Heegaard splittings recently given by the author and Hyam Rubinstein [6] because the Heegaard surface is weakly reducible. In fact, these mapping class groups combine behavior from two of the strongly irreducible classes: The automorphisms defined by the monodromy of the bundle are similar to the cyclic automorphisms induced by open book decompositions, which are further explored in [4]. The braid group elements, however, are reminiscent of the automorphisms induced by one-sided Heegaard splittings [5]. The present paper, however, uses completely different (and much more concise) techniques than those in [4] and [5], which also lead to the genus independent distance bound.

Recall that a Heegaard surface Σ is *weakly reducible* if there is a pair of disjoint compressing disks on opposite sides of Σ . These disks are called a *weak reducing pair*. In particular, the disks D^- , D^+ constructed above are a weak reducing pair for Σ . The proof of Theorem 1 is based on the following Lemma, which is a result of Masur-Schleimer's work on holes in the complex of curves:

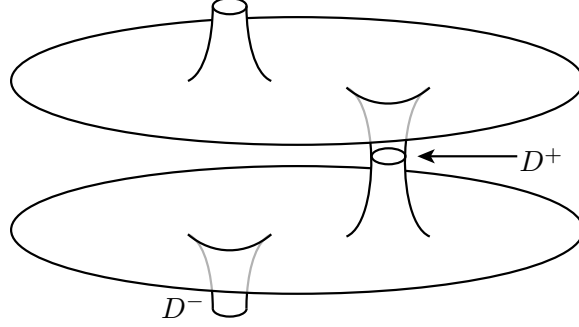
2. Lemma. *If the monodromy of π has displacement distance strictly greater than 28 then D^- , D^+ is the unique (up to isotopy) pair of weak reducing disks for Σ .*

Proof. When we attach tubes to construct Σ from F , we remove two disks from each component of the surface, then attach annuli to the boundaries where these disks were removed. Let P_1^\pm be the disks removed from F_1 and P_2^\pm the disks removed from F_2 such that the annulus A^+ along α_+ is attached to ∂P_1^+ and ∂P_2^+ , while the annulus A^- along α_- is attached to ∂P_1^- and ∂P_2^- .

As above, we let D^- , D^+ be the weak reducing disks dual to α_- , α_+ , respectively, shown in Figure 1. Assume for contradiction there is a second, distinct weak reducing pair E^- , E^+ , both with non-separating boundaries, labeled so that E^- is on the same side of Σ as D^- . Then one of the disks E^- , E^+ is distinct from the corresponding D^\pm and we will assume without loss of generality that E^- is not isotopic to D^- .

Isotope E^- to intersect D^- minimally. If we remove the disks P_1^+ , P_2^+ from F_1 and F_2 , then attach the annulus A^+ , the resulting surface bounds a handlebody B^- . If E^- is contained in B^- then let $C^- = E^-$. Otherwise, the complement $E^- \setminus B^-$ is a regular neighborhood of $E^- \cap D^-$ and we will let $C^- \subset E^-$ be an outermost disk of $E^- \cap B^-$.

Note that ∂C^- intersects $P_1^- \cup P_2^-$ in at most one arc. If ∂C^- is trivial in ∂B^- then it bounds a disk in ∂B^- containing P_1^- or P_2^- in its interior. If ∂C^- is isotopic to ∂P_1^- or ∂P_2^- in the complement of the two disks then E^- is isotopic to D^- , contradicting our initial assumption. If ∂C^- bounds a disk containing both P_1^- and P_2^- in its interior then ∂E^- is separating in Σ , which also contradicts our initial assumption. Thus C^- must intersect

FIGURE 1. The genus $2g + 1$ Heegaard surface.

P_1^- or P_2^- , and we will temporarily assume without loss of generality that it intersects P_1^- . This also implies that E^- must intersect ∂D^+ . Since E^+ is disjoint from E^- , we conclude that E^+ is not isotopic to D^+ .

If P_2^- is in the interior of a disk in ∂B^- bounded by ∂C^- then every arc of $\partial E^- \cap \partial B^-$ with one endpoint in ∂P_2^- must have its second endpoint in ∂P_1^- . This implies that $\partial E^- \cap P_1^-$ contains at least two more points than $\partial E^- \cap P_2^-$, contradicting the fact that both intersections contain the same number of points as $\partial E^- \cap \partial D^-$. We therefore conclude that ∂C^- is essential in ∂B^- .

The intersection $\partial C^- \cap F_1$ consists of arcs properly embedded in $F_1 \setminus P_1^+$, possibly with one arc intersecting P_1^- in a single subarc. We will think of the arcs as forming a graph G_1^- with vertices P_1^+ , P_1^- , and note that the vertex corresponding to P_1^- has valence zero or two.

If G_1^- is contained in a disk then there is an isotopy of ∂C^- in ∂B^- into $\partial B^- \cap F_2$. However, F_2 is incompressible in M and M is irreducible, so this implies that C^- is parallel into F_2 . Because C^- is disjoint from the annulus A^+ , it must be parallel into ∂B^- as well, which we ruled out above. Thus the graph G_1^- must contain an essential edge loop ℓ_1^- . Since G_1^- has two vertices, the edge length of ℓ_1^- is at most two. Similarly, we can find edge loops ℓ_1^+ in the graphs defined by $E^+ \cap F_1$ and ℓ_2^\pm in $E^\pm \cap F_2$.

Because E^- and E^+ are disjoint, any intersections between ℓ_1^+ and ℓ_1^- are contained in P_1^\pm , so the two loops intersect in at most two points. This implies there is an essential loop ℓ_1 disjoint from both. Similarly, there is an essential loop ℓ_2 disjoint from ℓ_2^+ and ℓ_2^- .

The handlebody B^- inherits an interval bundle structure, which defines a map $\gamma^+ : (F_1 \setminus P_1^+) \rightarrow (F_2 \setminus P_2^+)$. By Masur-Schleimer [8, Lemma 12.12], for any arcs β_1 of $C^- \cap F_1$ and β_2 of $C^- \cap F_2$, we have $d(\gamma^-(\beta_1), \beta_2) \leq 6$, where the distance is in the arc complex for $F_2 \setminus P_2^+$. (The arc complex is similar to the complex of curves, but with vertices representing properly embedded, essential arcs in a surface with boundary.) Because $F_2 \setminus P_2^+$ has a single puncture, every essential arc determines an essential loop in F_2 .

Disjoint arcs define essential loops that intersect in at most one point in F_2 , and thus have distance at most two in the curve complex $\mathcal{C}(F)$. Thus the distance from $\gamma^-(\ell_1^-)$ to ℓ_2^- is at most 12.

A similar argument applies to the map $\gamma^+ : (F_2 \setminus P_2^-) \rightarrow (F_1 \setminus P_1^-)$ defined by the surface that results from only attaching A^- to F . The image of ℓ_1^- under the composition $\gamma^+ \circ \gamma^-$ is the same as the image of ℓ_1^- under the monodromy map ϕ . By the argument above the distance in $\mathcal{C}(F)$ between these loops is at most $12 + 2 + 12 + 2 = 28$. Thus the displacement of ϕ is at most 28. Since we assumed that $d(\phi) > 28$, this contradiction implies that $E^- = D^-$ and $E^+ = D^+$ form the unique reducing pair for Σ . \square

Before we continue to the proof of Theorem 1, we note another implication related to the theory of topological index of surfaces. Recall that the *topological index* of Σ is zero if Σ is incompressible and otherwise is equal to the smallest value $(i + 1)$ such that the i th homotopy group of the disk set for Σ is non-trivial. If the disk complex is contractible then Σ will not have a well defined index.

McCullough has shown [9] that the set of compressing disks on one side of any two-sided embedded surface forms a contractible complex. The disk complex for Σ consists of these two contractible sets connected by a single edge between D^- and D^+ , so we have:

3. Corollary. *The disk set for Σ is contractible.*

Note that strongly irreducible Heegaard surfaces have index one. A number of weakly reducible Heegaard surface have been constructed with higher indices [1][7]. The surface constructed here appears to be the first class of minimal genus Heegaard surfaces known to have a contractible disk complex.

We now proceed to the main proof.

Proof of Theorem 1. Every automorphism $\psi \in \text{Mod}(M, \Sigma)$ takes the weak reducing pair D^-, D^+ onto a new weak reducing pair of disks. However by Lemma 2, D^-, D^+ is the only weak reducing pair so ψ either fixes these disks (setwise) or transposes them. The surface F results from compressing Σ along D^-, D^+ so we can choose a representative of ψ that takes F to itself. Moreover, because ψ fixes $F \cup \Sigma$ setwise, we can choose a representative that also fixes the arcs α_{\pm} .

By assumption, $\psi \in \text{Isot}(M, \Sigma)$ so ψ is isotopic to the identity on M and defines an isotopy from F to itself. The surface F consists of two leaves of the surface bundle defined by π , each of which has injective fundamental group. The isotopy thus defines an injection from a semi-direct product $\pi_1(F_1) \rtimes \mathbf{Z}$ into $\pi_1(M)$. As in Theorem 11.1 in [3], this implies that the isotopy is induced by a surface bundle structure $\pi' : M \rightarrow S^1$. Because the fiber of π' is F_1 , which is also the fiber of π , we conclude that π' is isotopic to π so the isotopy ψ must be defined by the maps $\{r_t\}$ that spin F around the monodromy of the surface bundle.

Each ψ thus determines a half integer and we find a homomorphism from $Isot(M, \Sigma)$ to \mathbf{Z} (or rather to the half integers). If ψ is in the kernel \mathcal{K} of this homomorphism then the restriction of ψ to F is isotopic to the identity (ignoring the endpoints of α_{\pm}). If we consider the endpoints, however, we find a homomorphism from \mathcal{K} to the braid group B_{2, F_1} . The kernel \mathcal{K}' of this second homomorphism is the set of all isotopies of $F \cup \alpha_{\pm}$ that fix the endpoints of α_{\pm} in F_1 . The paths of the endpoints in F_2 , in turn, define a homomorphism into the two-strand braid group on F_2 . The reader can check that these braids are all homotopy trivial in F_2 and that all homotopy trivial braids arise in this way.

The kernel \mathcal{K}'' of this final homomorphism is the set of isotopies of the arcs that fix all four endpoints. Any automorphism of Σ induced in this way would be the identity outside a pair of annuli. However, there is a canonical vertical arc transverse to each annulus, so no non-trivial automorphism can arise in this way. Thus \mathcal{K}'' is trivial so \mathcal{K}' is isomorphic to the homotopy trivial subgroup of B_{2, F_2} . By construction, \mathcal{K} is an extension of B_{2, F_1} by $\mathcal{K}' = B_{2, F_2}$ so $\mathcal{K} = \mathcal{B}$. Finally, $Isot(M, \Sigma)$ is an extension of \mathcal{K} by the integers, with \mathcal{K} a normal subgroup and the integers acting on \mathcal{K} by conjugating by the monodromy map ϕ . Thus $Isot(M, \Sigma)$ is the semi-direct product \mathcal{A} defined above. \square

REFERENCES

1. David Bachman and Jesse Johnson, *On the existence of high index topologically minimal surfaces*, Math. Res. Lett. **17** (2010), no. 3, 389–394. MR 2653676 (2011e:57025)
2. David Bachman and Saul Schleimer, *Surface bundles versus Heegaard splittings*, Comm. Anal. Geom. **13** (2005), no. 5, 903–928. MR 2216145 (2006m:57027)
3. John Hempel, *3-Manifolds*, Princeton University Press, Princeton, N. J., 1976, Ann. of Math. Studies, No. 86. MR 0415619 (54 #3702)
4. Jesse Johnson, *Heegaard splittings and open books*, preprint (2011), arXiv:1110.2142.
5. ———, *One-sided and two-sided Heegaard splittings*, preprint (2011), arXiv:1112.0471.
6. Jesse Johnson and Hyam Rubinstein, *Mapping class groups of Heegaard splittings*, preprint (2007), arXiv:math/0701119.
7. Jung Hoon Lee, *Critical Heegaard surfaces obtained by amalgamation*, preprint (2011), arXiv:1109.3862.
8. Howard Masur and Saul Schleimer, *The geometry of the disk complex*, preprint (2010), arXiv:1010.3174.
9. Darryl McCullough, *Virtually geometrically finite mapping class groups of 3-manifolds*, J. Differential Geom. **33** (1991), no. 1, 1–65. MR 1085134 (92c:57001)

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